QUANTUM ERGODICITY FOR PRODUCTS OF HYPERBOLIC PLANES

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Abstract. For manifolds with geodesic flow that is ergodic on the unit tangent bundle, the quantum ergodicity theorem implies that almost all Laplacian eigenfunctions become equidistributed as the eigenvalue goes to infinity. For a locally symmetric space with a universal cover that is a product of several upper half planes, the geodesic flow has constants of motion so it can not be ergodic. It is, however, ergodic when restricted to the submanifolds defined by these constants. In accordance, we show that almost all eigenfunctions become equidistributed on these submanifolds.

INTRODUCTION

The Quantum Ergodicity Theorem [2, 14, 16], is a celebrated result concerning the behavior of Laplacian eigenfunctions on compact manifolds with an ergodic geodesic flow, stating that most eigenfunctions become equidistributed on the unit tangent bundle with respect to the volume measure. We currently lack a general understanding of the situation when the geodesic flow is not ergodic (but still not integrable). In this paper we look at a special example, that of a locally symmetric space with a universal cover that is a product of upper half planes. The geodesic flow on this space is no longer ergodic, yet, it does posses some chaotic features, suggesting that Laplacian eigenfunctions still become equidistributed (on the correct space).

In the special case of one half plane, the geodesic flow is ergodic, so that the Quantum Ergodicity Theorem applies. In fact, in this case it is believed that a much stronger result holds, that is, that all eigenfunctions become equidistributed as the eigenvalue goes to infinity. This notion is referred to as *Quantum Unique Ergodicity* and is conjectured to hold for surfaces of negative curvature [9]. Perhaps the strongest evidence for the QUE conjecture comes from the analysis on arithmetic surfaces, i.e., $X = \Gamma\backslash\mathbb{H}$ with Γ a congruence subgroup. For arithmetic

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surfaces there are additional symmetries, Hecke operators commuting with each other and with the Laplacian. A joint eigenfunction of the Laplacian and all Hecke operators is called a Hecke eigenfunction. In [8], Lindenstrauss showed that indeed, for any sequence of Hecke eigenfunctions the corresponding quantum measures converge to the volume measure. We note that by Watson's formula for triple integrals [15], the Grand Riemann Hypothesis implies QUE for Hecke eigenfunctions with an effective rate of convergence.

We now proceed to the high rank case and consider the locally symmetric space $X = \Gamma\backslash\mathcal{H}$, with $\mathcal{H} = \mathbb{H}\times\cdots\times\mathbb{H}$ a product of d hyperbolic planes, and Γ an irreducible co-compact lattice in $\mathcal{G} = \mathrm{PSL}(2,\mathbb{R})^d$. We note that in this case the geodesic flow (on the tangent bundle) has $d > 1$ independent constants of motion given by the partial energy functions $E_j(z,\xi) = \|\xi_j\|^2_{z_j}$. Consequently, the geodesic flow can not be ergodic on the unit tangent bundle. It is, however, ergodic when restricted to the generalized energy shells

$$\Sigma(\mathbf{E}) = \{(z,\xi) \in TX | E_j(z,\xi) = E_j\} \subset SX,$$

for any level $\mathbf{E} \in [0,1]^d$ with $\sum_j E_j = 1$ (we normalize the energy so that the energy shell lies in SX). Notice, that the structure and the dynamics on each energy shell is determined by the set of singularities $\{j|E_j = 0\}$, so that any two energy shells with the same singularities can be identified.

The algebra of invariant differential operators in this case is generated by d partial Laplacians $\triangle_j = y_j^2(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2})$ acting on each hyperbolic plane. Any Laplacian eigenfunction ϕ_k (that is, a joint eigenfunction of all the partial Laplacians $\triangle_j\phi_k + \lambda_{k,j}\phi_k = 0$), can be interpreted as a distribution on the unit tangent bundle SX (via a corresponding Wigner distribution). This distribution is concentrated on a corresponding energy shell $\Sigma(\mathbf{E}_k) \subseteq SX$, with $\mathbf{E}_k = \frac{(\lambda_{k,1},\ldots,\lambda_{k,d})}{\lambda_{k,1}+\cdots+\lambda_{k,d}}$. The projection of this distribution to the base manifold X is the measure defined by the density $d\mu_k = |\phi_k|^2 dz$.

In this paper we show an analogous result to the Quantum Ergodicity Theorem in this setting. For every energy level $\mathbf{E}$ the flow on the energy shell $\Sigma(\mathbf{E})$ is ergodic. Correspondingly, we show that for almost any sequence of eigenfunctions, ϕ_k, with (normalized) eigenvalues $\frac{(\lambda_{k,1},\ldots,\lambda_{k,d})}{\lambda_{k,1}+\cdots+\lambda_{k,d}} = \mathbf{E}_k \to \mathbf{E}$, the corresponding distributions converge to the volume measure of the energy shell $\Sigma(\mathbf{E})$. Note that the projection to the base of the volume measure from any energy shell is the volume measure of X, hence, the above result implies that for almost any

sequence of eigenfunctions, the measures μ_k converge to the volume measure of X.

Remark 0.1. It is reasonable that similar results could be proved more generally with pseudodifferential calculus. See [18] for analogous results on reduced quantum ergodicity in the presence of symmetries.

The above Quantum Ergodicity result holds for any irreducible co-compact lattice and for any orthonormal basis of Laplacian eigenfunctions (without taking into account action of Hecke operators). We note that, as in rank one, much more is known in the arithmetic setting. When Γ is a congruence subgroup (coming from a Quaternion algebra over a corresponding number field), there are Hecke operators acting on $L^2(\Gamma\backslash\mathcal{H})$ commuting with all the partial Laplacians. If one considers Hecke eigenfunctions then it is likely that, again, the only limiting measure on X obtained as a quantum limit (respectively its lift to $\Gamma\backslash\mathcal{G}$), is the volume measure[1]. In particular, this would imply that for **any** sequence of Hecke eigenfunctions, with eigenvalues $\mathbf{E}_k \to \mathbf{E}$, the corresponding distributions converge to the volume measure of $\Sigma(\mathbf{E})$.

Remark 0.2. In [12], Silberman and Venkatesh generalized the lift of the limiting measures to the more general setting of higher rank locally symmetric spaces $\Gamma\backslash\mathcal{G}/K$ with $\mathcal{G}$ a semi-simple connected Lie group. In [13], for the special case of $\mathcal{G} = \mathrm{PGL}(d,\mathbb{R})$ with d prime, they used this lift to generalize the results of [8], and show that for any sequence of (non-degenerate) Hecke eigenfunctions, the limiting measure is the Haar measure.

Results. Let $X = \Gamma\backslash\mathcal{H}$, with $\mathcal{H} = \mathbb{H}\times\cdots\times\mathbb{H}$ a product of d hyperbolic planes, and Γ an irreducible co-compact lattice in $\mathcal{G} = \mathrm{PSL}(2,\mathbb{R})^d$. Let $\{\phi_k\}$ be an orthonormal basis of $L^2(X)$ consisting of (joint) Laplacian eigenfunctions $(\triangle_j + \lambda_{k,j})\phi_k = 0$, $\lambda_{k,j} = \frac{1}{4} + r_{k,j}^2$. For any $\mathbf{L} = (L_1,\ldots,L_d) \in [\frac{1}{2},\infty)^d$, let

$$\mathcal{I}(\mathbf{L}) = \{k\colon\ \|r_k - \mathbf{L}\|_\infty \leq 1/2\}$$

denote the set of eigenfunctions with eigenvalues in a window around $\mathbf{L}$, and denote by $N(\mathbf{L}) = \sharp\mathcal{I}(\mathbf{L})$ the number of such eigenfunctions.

Remark 0.3. The choice for the window, $\mathcal{I}(\mathbf{L})$, to be of volume one in $\mathbb{R}^d$ is mainly cosmetic. The same results (with essentially the same proofs) also holds if we take $\mathcal{I}(\mathbf{L})$ to be a window of any given size.

[1] As pointed out by Elon Lindenstrauss, such a result should follow from the same arguments applied in [1, 7, 8].

For any $\mathbf{E} \in [0,1]^d$ (with $\sum E_j = 1$) we can identify the generalized energy shells, $\Sigma(\mathbf{E})$, with the quotients $\Gamma\backslash\mathcal{G}/\prod_{E_j=0} K_j$. So for instance when $d = 2$ and $\mathbf{E} = (1,0)$ the energy shell $\Sigma(1,0)$ is identified with $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})\times\mathbb{H}$. Under this identification the volume measure of $\Sigma(\mathbf{E})$ is (up to normalization) the measure induced from the Haar measure of $\mathcal{G}$.

To each eigenfunction ϕ_k we attach a distribution S_{ϕ_k} on $\Gamma\backslash\mathcal{G}$ (coming from the Wigner distribution on TX), that coincides with μ_k on $\mathcal{K}$-invariant function (see sections 3). If we take $\mathbf{L} \to \infty$ so that $\frac{(L_1^2,\dots,L_d^2)}{\|\mathbf{L}\|^2} \to \mathbf{E}$, then for $k \in \mathcal{I}(\mathbf{L})$ the normalized eigenvalues $\mathbf{E}_k \sim \mathbf{E}$ and the distributions S_{ϕ_k} become close to probability measures on $\Sigma(\mathbf{E})$. Given a smooth test function $a \in C^\infty(\Sigma(\mathbf{E}))$, we evaluate how far are these distributions (equivalently measures) from the volume measure. We first show that when (at least one of) the eigenvalues go to infinity, on average, the distributions S_{ϕ_k} converge to the volume measure on $\Gamma\backslash\mathcal{G}$.

Theorem 1. *For any $a \in C^\infty(\Gamma\backslash\mathcal{G})$,*

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})} \sum_{k\in\mathcal{I}(L)} S_{\phi_k}(a) = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathcal{G})} \int_{\Gamma\backslash\mathcal{G}} a(g)dg.$$

Note that this theorem holds for any smooth function on $\Gamma\backslash\mathcal{G}$, and any limit $\mathbf{L} \to \infty$. In particular, if we start from $a \in C^\infty(\Sigma(\mathbf{E}))$ and take $\mathbf{L} \to \infty$ with $\frac{(L_1^2,\dots,L_d^2)}{\|\mathbf{L}\|^2} \to \mathbf{E}$, we get that on average

$$\frac{1}{N(\mathbf{L})} \sum_{k\in\mathcal{I}(L)} S_{\phi_k}(a) \to \int_{\Sigma(\mathbf{E})} a d\mathrm{vol}.$$

Next, we study the variation from the average. To do this, define the variance of the distributions S_{ϕ_k}, $k \in \mathcal{I}(\mathbf{L})$ (with respect to the test function $a \in C^\infty(\Sigma(\mathbf{E}))$ as

$$\mathrm{Var}_{\mathbf{L}}(a) = \frac{1}{N(\mathbf{L})} \sum_{k\in\mathcal{I}(\mathbf{L})} \left| S_{\phi_k}(a) - \int_{\Sigma(\mathbf{E})} a d\mathrm{vol} \right|^2.$$

Theorem 2. *For any $a \in C^\infty(\Sigma(\mathbf{E}))$, as $\mathbf{L} \to \infty$ with $\frac{(L_1^2,\dots,L_d^2)}{\|\mathbf{L}\|^2} \to \mathbf{E}$,*

$$\lim_{\|\mathbf{L}\|\to\infty} \mathrm{Var}_{\mathbf{L}}(a) = 0.$$

In particular, using a diagonalization argument (see [16]), this variance estimate implies that for almost any sequence of ϕ_k, with normalized eigenvalues converging to some energy $\mathbf{E}$, the distributions S_{ϕ_k} converge to the volume measure of the corresponding energy shell.

Moreover, since the projection of the volume measure of any energy shell to X is the volume measure of X, we get the following corollary:

Corollary 1. *For any $a \in C^\infty(X)$, as $\mathbf{L} \to \infty$*

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})} \sum_{k\in\mathcal{I}(\mathbf{L})} \left| \int_X a(z)|\phi_k(z)|^2 dz - \int_X a(z)dz \right|^2 = 0.$$

This implies that almost all of the measures μ_k converge to the volume measure of X.

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1. Background and Notation

1.1. **The hyperbolic plane.** Let $\mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ denote the upper half plane. This is a symmetric space with group of isometries $G = \mathrm{PSL}(2,\mathbb{R})$ (acting by linear transformations). Let $K \subseteq G$ be the stabilizer of $i \in \mathbb{H}$ and $P \subseteq G$ be the stabilizer of $\infty \in \partial\mathbb{H}$. We use coordinates corresponding to the identification of $\mathbb{H} = G/K \cong P$. For $z = x + iy \in \mathbb{H}$ and $\theta \in [0,\pi)$ we let $p_z = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{1/y} \end{pmatrix} \in P$ and $k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K$. In these coordinates the (normalized) Haar measures of P and K are given by $dp = dz = \frac{dxdy}{y^2}$ and $dk = \frac{d\theta}{\pi}$. The Haar measure of G is then $dg = dpdk$.

The Lie algebra $\mathrm{sl}(2,\mathbb{R})$, is generated by $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In the (x, y, θ) coordinates these elements are given by the following differential operators (c.f. [6, Chapter IV

§4]):

$$\begin{aligned}
W &= \frac{\partial}{\partial\theta}\\
H &= -2y\sin(2\theta)\frac{\partial}{\partial x}+2y\cos(2\theta)\frac{\partial}{\partial y}+\sin(2\theta)\frac{\partial}{\partial\theta}\\
X &= y\cos(2\theta)\frac{\partial}{\partial x}+y\sin(2\theta)\frac{\partial}{\partial y}+\sin^2(\theta)\frac{\partial}{\partial\theta}
\end{aligned}$$

We consider the function $E(z,\xi)=\|\xi\|_z^2=\frac{\xi_x^2+\xi_y^2}{y^2}$ as an energy function. For any $E>0$ we can identify the energy shell

$$T\mathbb{H}_E=\{(z,\xi)\colon E(z,\xi)=E\}$$

with G (and also with the unit tangent bundle $T\mathbb{H}_1$) via the map $g(z,\xi)=p_zk_\theta$ with $\theta=\tan^{-1}(\xi_y/\xi_x)$. The zero section $T\mathbb{H}_0\cong\mathbb{H}$ is identified with G/K. The energy shells are invariant under the geodesic flow, that (under the above identification) is given by the action of $A(t)=e^{Ht}=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix}$ on G from the right.

1.2. **Products of hyperbolic planes.** Let $\mathcal{H}=\mathbb{H}\times\cdots\times\mathbb{H}$ be a product of d hyperbolic planes. This is a symmetric space with group of isometries $\mathcal{G}=\prod_{j=1}^d G_j$ (where $G_j=G=\mathrm{PSL}(2,\mathbb{R})$). We have a decomposition $\mathcal{G}=\mathcal{P}\mathcal{K}$, where $\mathcal{K}=\prod K_j$ and $\mathcal{P}=\prod P_j$. For $z=(z_1,\dots,z_d)\in\mathcal{H}$ and $\theta=(\theta_1,\dots,\theta_d)\in[0,\pi)^d$ we let $p_z=(p_{z_1},\dots,p_{z_d})\in\mathcal{P}$ and $k_\theta=(k_{\theta_1},\dots,k_{\theta_d})\in\mathcal{K}$. The Haar measures of $\mathcal{P},\mathcal{K}$, and $\mathcal{G}$ are then $dp=dp_1\cdots dp_d$, $dk=dk_1\cdots dk_d$ and $dg=dg_1\cdots dg_d$ respectively. We will denote by W_j,H_j and X_j the action of the corresponding differential operator on the j'th factor. For every $\mathbf{E}=(E_1,\dots,E_d)\in[0,\infty)^d$ we can identify $\mathcal{G}/\prod_{E_j=0}K_j$ with a corresponding energy shell inside the tangent bundle $T\mathcal{H}$. The geodesic flow through the point $(g,\mathbf{E})=((g_1,\dots,g_d),(E_1,\dots,E_d))\in\mathcal{G}\times[0,\infty)^d$ is then given by the right action of $A_{\mathbf{E}}(t)=\prod_j A_j(\sqrt{E}_jt)$ (where each $A_j(t)=e^{tH_j}$ is the diagonal action on the corresponding factor).

1.3. **Irreducible lattices.** A discrete subgroup $\Gamma\subset\mathcal{G}$ is called a lattice if the quotient $\Gamma\backslash\mathcal{G}$ has finite volume, and co-compact when $\Gamma\backslash\mathcal{G}$ is compact. We say that a lattice $\Gamma\subset\mathcal{G}$ is irreducible, if for every (non-central) normal subgroup $N\subset\mathcal{G}$ the projection of Γ to $\mathcal{G}/N$ is dense. An equivalent condition for irreducibility, is that for any nontrivial $1\neq\gamma\in\Gamma$, none of the projections $\gamma_j\in G_j$ are trivial [11, Theorem 2]. Examples of irreducible (co-compact) lattices can be constructed from

norm one elements of orders inside an appropriate quaternion algebra over a totally real number field [10]. In fact when $d \geq 2$, Margulis's arithmeticity theorem states that, up to commensurability, these are the only examples.

Let $\Gamma \subset \mathcal{G}$ be an irreducible co-compact lattice, we now go over the classification of the different elements of Γ (see e.g., [4, 11]). Recall that an element $g_j \in G_j = \mathrm{PSL}(2,\mathbb{R})$ is called hyperbolic if $|\mathrm{Tr}(g_j)| > 2$, elliptic if $|\mathrm{Tr}(g_j)| < 2$, and parabolic if $|\mathrm{Tr}(g_j)| = 2$. Then for any nontrivial $1 \neq \gamma \in \Gamma$, the projections to the different factors are either hyperbolic or elliptic. There are purely hyperbolic elements (where all projections are hyperbolic), and mixed elements (where some projections are hyperbolic and other elliptic). There could also be a finite number of torsion points that are purely elliptic.

1.4. **Spectral decomposition.** Let Γ be an irreducible co-compact lattice in $\mathcal{G}$, so that $X = \Gamma\backslash\mathcal{H}$ is a compact Riemannian manifold. The algebra of invariant differential operators is generated by the d partial hyperbolic Laplacians $\triangle_j = y_j^2(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2})$. The space $L^2(X)$ has an orthonormal basis, $\{\phi_k\}$, composed of Laplacian eigenfunctions, $(\triangle_j + \lambda_{k,j})\phi_k = 0$, with all partial eigenvalues $\lambda_{k,j} \geq 0$ (in fact $\lambda_{k,j} > 0$ unless $\phi_k \equiv 1$). For each eigenfunction ϕ_k, we think of the normalized eigenvalues $\mathbf{E}_k = \frac{(\lambda_{k,1},\dots,\lambda_{k,d})}{\lambda_{k,1}+\dots+\lambda_{k,d}}$ as a quantum energy level. We use the standard parametrization $\lambda_{k,j} = r_{k,j}^2 + \frac{1}{4}$ with $r_{k,j} \geq 0$ for $\lambda_{k,j} \geq \frac{1}{4}$ and $r_{k,j} \in i(0,\frac{1}{2})$ otherwise.

In the case where some of the eigenvalues are small, $\lambda_{k,j} < \frac{1}{4}$, we say that ϕ_k is exceptional. We note that in this setting there could be infinitely many exceptional eigenfunctions (in contrast to the rank one case where there could be only finitely many). When Γ is a congruence subgroup, it is conjectured that there are no exceptional eigenfunctions at all. However, in our general setting the most we can say is that the exceptional eigenfunctions are of density zero (Lemma A.3).

1.5. **Fourier decomposition.** Consider the homogeneous space $Y = \Gamma\backslash\mathcal{G}$. We can identify our space $X = \Gamma\backslash\mathcal{H}$ with the double quotient $\Gamma\backslash\mathcal{G}/\mathcal{K} = Y/\mathcal{K}$, and think of functions on X as $\mathcal{K}$-invariant functions on Y. For $n \in \mathbb{Z}^d$, let $\mathcal{F}_n(Y)$ denote the joint eigenspaces for W_j with eigenvalues $2in_j$ respectively. That is

$$\mathcal{F}_n(Y) = \left\{f \in C^\infty(Y) | f(p_z k_\theta) = e^{2i(n\cdot\theta)} f(p_z)\right\}.$$

Any $a \in C^\infty(Y)$ has a $\mathcal{K}$-Fourier decomposition $a = \sum_{n\in\mathbb{Z}^d} a_n$ with $a_n \in \mathcal{F}_n(Y)$. For any integer $s \geq 1$, the functions a_n in this decomposition are uniformly bounded $\|a_n\|_\infty \leq \frac{\|W_j^s a\|_\infty}{|2n_j|^s}$ for any $1 \leq j \leq d$.

Consider the operators $E_j^\pm = H_j \pm i(2X_j - W_j)$. These operators satisfy $[W_j, E_j^\pm] = 2iE_j^\pm$, so that they act as raising an lowering operators (i.e., $E_j^\pm : \mathcal{F}_n(Y) \to \mathcal{F}_{n\pm e_j}(Y)$). Furthermore, on K_j invariant functions the action of $E_j^- E_j^+$ coincides with the action of (4 times) the j'th partial Laplacian.

1.6. **Reduced Ergodicity.** The geodesic flow on TX is the flow induced from the geodesic flow on $T\mathcal{H}$. The unit tangent bundle,

$$SX = \left\{(z,\xi) \in X | \sum E_j(z_j,\xi_j) = 1\right\},$$

is invariant under this flow. However, in contrast to the rank one case, the flow on SX is no longer ergodic (because the functions $E_j(z_j,\xi_j)$ are d independent constants of motion). Instead of the unit tangent bundle, for any $\mathbf{E} = (E_1,\dots,E_d) \in [0,\infty)^d$ we consider a generalized energy shell

$$\Sigma(\mathbf{E}) = \{(z,\xi) \in TX | \forall j,\ E_j(z_j,\xi_j) = E_j\}.$$

We can naturally identify these energy shells $\Sigma(\mathbf{E}) \cong \Gamma\backslash\mathcal{G}/\prod_{E_j=0} K_j$, and think of functions on $\Sigma(\mathbf{E})$ as $\prod_{E_j=0} K_j$ invariant functions on Y. By Moore's ergodicity theorem [19, Theorem 2.2.6], the geodesic flow restricted to $\Sigma(\mathbf{E})$ is ergodic (with respect to the volume measure on $\Sigma(\mathbf{E})$). In fact, each one of the flows $A_j(t) = e^{H_j t}$ is already ergodic on Y. In particular, for any $a \in C^\infty(Y)$ let

$$\langle a\rangle_{\mathbf{E}}^T = \frac{1}{T}\int_0^T a \circ A_{\mathbf{E}}(t)dt,$$

denote the time average of a with respect to the geodesic flow $A_{\mathbf{E}}(t) = \prod_j A_j(\sqrt{E_j}t)$ on Y. Then as $T \to \infty$, this time averages converges, in $L^2(Y)$, to the phase space average

$$\langle a\rangle_{\mathbf{E}}^T \xrightarrow{L^2(Y)} \int_Y a(g)dg.$$

Note that this property is stronger then ergodicity on $\Sigma(\mathbf{E})$, as we do not assume that a is K_j invariant when $E_j = 0$.

1.7. **Notations.** We make use of the following notation: Given a positive function $g(x)$, we denote $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a constant $c > 0$ such that $\forall x,\ |f(x)| \leq cg(x)$. If the constant depends on some parameters, say ϵ, δ, then they will appear as a subscript $f(x) = O_{\epsilon,\delta}(g(x))$ or $f(x) \ll_{\epsilon,\delta} g(x)$. We also use the notation $f(x) = o(g(x))$ meaning that $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$.

2. Outline of proof

We now describe the outline for the proof of the main result (Theorem 2). For each eigenfunction ϕ_k, we identify the corresponding Wigner distribution with a distribution S_{ϕ_k} on $Y = \Gamma\backslash\mathcal{G}$. On $\mathcal{K}$-invariant functions $a \in C^\infty(\Gamma\backslash\mathcal{G}/\mathcal{K}) = C^\infty(X)$ these distributions coincide with the quantum measures,

$$S_{\phi_k}(a) = \int_X a(z)|\phi_k(z)|^2 dz.$$

We will show that the distributions S_{ϕ_k} satisfy the following properties:
1.(Invariance) For $j = 1, \dots, d$ consider the ergodic flows $A_j(t)$ on Y (given by the right action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the j'th factor). Then, as $r_{k,j} \to \infty$ the distributions S_{ϕ_k} becomes invariant in the sense that for $a \in C^\infty(Y)$,

$$|S_k(a \circ A_j(t)) - S_k(a)| \ll_{a,t} \frac{1}{r_{k,j}}.$$

2.(Positivity) When the normalized eigenvalue $\mathbf{E}_k$ is close to an energy level $\mathbf{E}$, the distribution S_{ϕ_k} is close to positive measure on $\Sigma(\mathbf{E})$. More precisely, for every eigenfunction ϕ_k, and any $\mathcal{J} \subseteq \{1, \dots, d\}$ there is normalized function $\Phi_{k,\mathcal{J}} \in L^2(Y)$, such that for functions $a \in C^\infty(\Gamma\backslash\mathcal{G}/\prod_{j\notin\mathcal{J}} K_j)$

$$S_{\phi_k}(a) = \langle a\Phi_{k,\mathcal{J}}, \Phi_{k,\mathcal{J}}\rangle + O_a(R_{k,\mathcal{J}}^{-1/4}),$$

with $R_{k,\mathcal{J}} = \min_{j\in\mathcal{J}} r_{k,j}$.
3.(Local Weyl's law) The last property, stated in Theorem 1, deals with the average of the distributions S_{ϕ_k} over the window $\mathcal{I}(\mathbf{L})$. That is, for $a \in C^\infty(Y)$,

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})} \sum_{k\in\mathcal{I}(\mathbf{L})} S_k(a) = \frac{1}{\mathrm{vol}(Y)} \int_Y a(g)dg.$$

Assuming these properties are satisfied the proof goes as follows:

Proof of Theorem 2. Fix $\mathbf{E} \in [0,1)^d$ (with $\sum_j E_j = 1$) and assume that $\mathbf{L} \to \infty$ with $\frac{(L_1^2,\dots,L_d^2)}{\|\mathbf{L}\|^2} \to \mathbf{E}$. If we set $\mathcal{J} = \{j | E_j \neq 0\}$, then for any $j \in \mathcal{J}$ we have $L_j \gg \|\mathbf{L}\|$. Fix a smooth function $a \in C^\infty(\Sigma(\mathbf{E}))$ and assume $\int_{\Sigma(\mathbf{E})} a d\mathrm{vol} = 0$. Identify $\Sigma(\mathbf{E}) = \Gamma \backslash \mathcal{G} / \prod_{i \notin \mathcal{J}} K_i$, and think of a as a function in $C^\infty(Y)$ invariant under $\prod_{i\notin\mathcal{J}} K_i$. Now consider the "time average" with respect to the geodesic flow $A_{\mathbf{E}}(t) = \prod_{j\in\mathcal{J}} A(\sqrt{E}_j t)$,

$$\langle a\rangle_{\mathbf{E}}^T = \frac{1}{T}\int_0^T a \circ A_{\mathbf{E}}(t)dt.$$

For any $k \in \mathcal{I}(\mathbf{L})$ and $j \in \mathcal{J}$, the distribution S_{ϕ_k} is $A_j(t)$ invariant (up to $O(\frac{1}{\|\mathbf{L}\|})$). Hence, for $k \in \mathcal{I}(\mathbf{L})$

$$S_{\phi_k}(\langle a\rangle_{\mathbf{E}}^T) = S_{\phi_k}(a) + O_{a,T}(\frac{1}{\|\mathbf{L}\|}).$$

Consequently,

$$\frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(\mathbf{L})} |S_{\phi_k}(a)|^2 = \frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(\mathbf{L})} |S_{\phi_k}(\langle a\rangle_j^T)|^2 + O_{a,T}(\frac{1}{\|\mathbf{L}\|}).$$

Since the action of $A_j(t),\ j \in \mathcal{J}$ commutes with the action of K_i for any $i \notin \mathcal{J}$ the time average $\langle a\rangle_{\mathbf{E}}^T$ remains invariant under $\prod_{i\notin\mathcal{J}} K_i$. We can use this invariance to show for any $k \in \mathcal{I}(\mathbf{L})$

$$|S_{\phi_k}(\langle a\rangle_{\mathbf{E}}^T)|^2 \le S_{\phi_k}(|\langle a\rangle_{\mathbf{E}}^T|^2) + O_{a,T}(\|\mathbf{L}\|^{-1/4}).$$

Indeed, by the positivity property,

$$|S_{\phi_k}(\langle a\rangle_{\mathbf{E}}^T)|^2 = |\langle\langle a\rangle_{\mathbf{E}}^T \Phi_{k,\mathcal{J}}, \Phi_{k,\mathcal{J}}\rangle|^2 + O_{a,T}(\|\mathbf{L}\|^{-1/4}),$$

and by Cauchy-Schwartz

$$|\langle\langle a\rangle_{\mathbf{E}}^T \Phi_{k,j}, \Phi_{k,j}\rangle|^2 \le \langle|\langle a\rangle_{\mathbf{E}}^T|^2 \Phi_{k,j}, \Phi_{k,j}\rangle = S_{\phi_k}(|\langle a\rangle_{\mathbf{E}}^T|^2) + O_{a,T}(\|\mathbf{L}\|^{-1/4}).$$

Consequently, for the average also

$$\frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(\mathbf{L})} |S_{\phi_k}(a)|^2 \le \frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(\mathbf{L})} S_{\phi_k}(|\langle a\rangle_{\mathbf{E}}^T|^2) + O_{a,T}(\|\mathbf{L}\|^{-1/4}).$$

Taking $\|\mathbf{L}\| \to \infty$ the local Weyl's law then implies

$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(\mathbf{L})} |S_{\phi_k}(a)|^2 \le \frac{1}{\mathrm{vol}(Y)}\int_Y |\langle a\rangle_{\mathbf{E}}^T|^2 dg.$$

Finally, the ergodicity of the flows imply that in the limit $T \to \infty$,

$$\langle a\rangle_{\mathbf{E}}^T \xrightarrow{L^2(Y)} \int_Y a(g)dg = \int_{\Sigma(\mathbf{E})} a d\mathrm{vol} = 0,$$

concluding the proof. □

It thus remains to verify that the distributions S_{ϕ_k} indeed satisfy the desired properties. In section 3 we follow the arguments used by Lindenstrauss in [7] to show the invariance and positivity properties. Then in sections 4,5 we will follow Zelditch's formalism [16] for the Wigner distribution via the Helgason-Fourier transform to give a local Weyl's law.

3. Micro Local Lift

In this section we recall the construction of [7, 16], lifting a quantum measure μ_{ϕ_k} on X to a distribution S_{ϕ_k} on Y. We then verify that these distributions satisfy the desired properties of invariance and positivity. This is essentially the content of [7, Theorem 4.1] and [7, Theorem 3.1], however, since the formulation we need is slightly different we will include the proofs. Throughout this section the eigenfunction $\phi = \phi_k$ is fixed, and for notational convenience the subscript k will be omitted.

3.1. **Definition.** For $r = (r_1, \dots, r_d)$ let $\phi \in C^\infty(X) \equiv \mathcal{F}_0(Y)$ be a joint eigenfunction of all the $\triangle_j$'s with eigenvalues $\lambda_j = (\frac{1}{4} + r_j^2)$ respectively. We construct from ϕ by induction a sequence of functions $\phi_n \in \mathcal{F}_n(Y),\ n \in \mathbb{Z}^d$: Let $\phi_0(g) = \phi(g(i))$, and define

$$\phi_{n \pm e_j} = \frac{1}{2ir_j + 1 \pm 2n_j} E_j^{\pm} \phi_n \tag{3.1}$$

Definition 3.1. Define distribution S_ϕ on $C^\infty(Y)$ by

$$S_\phi(a) = \lim_{N \to \infty} \langle a \sum_{\|n\|_\infty \le N} \phi_n, \phi_0 \rangle_Y,$$

where $\langle a, b \rangle_Y = \int_Y a(g)\bar{b}(g)dg$.

Note that the rapid decay of $\|a_n\|_\infty$ as $\|n\| \to \infty$, imply that the sum absolutely converges and the distributions $S_\phi(a)$ are bounded by

$$\sum_n \|a_n\|_\infty \ll \max_j \left\| W_j^{2d} a \right\|_\infty .$$

Remark 3.1. This definition coincides with the Wigner distribution constructed by Zelditch [16] via Helgason's Fourier calculus (see section 4.2). Also, see [12] for a representation theoretic interpretation of this construction that is more natural when generalizing it to locally symmetric spaces.

3.2. **Invariance.** Recall the family of one parameter (ergodic) flows, $A_j(t) = e^{H_j t}$, $1 \leq j \leq d$ on Y. We now show that when the j'th eigenvalue $\lambda_j = (\frac{1}{4} + r_j^2)$ becomes large the distribution S_ϕ becomes invariant under the corresponding flow $A_j(t)$ (c.f., [7, Theorem 4.1])

Proposition 3.2. *For fixed $a \in C^\infty(Y)$,*

$$|S_\phi(a) - S_\phi(a \circ A_j(t))| \ll_{a,t} \frac{1}{r_j}$$

Proof. The flow A_j is generated by $H_j \in \mathrm{sl}_2(\mathbb{R})$ in the sense that $\frac{d}{dt}(a \circ A_j(t)) = H_j(a \circ A_j(t))$. Let $F(t) = S_\phi(a \circ A_j(t))$, then its derivative is given by $F'(t) = S_\phi(H_j(a \circ A_j(t))$.

Now use the differential equation [7, Proposition 4.2]

$$S_\phi((4ir_j H_j + H_j^2 + 4X_j^2)a) = 0,$$

to deduce

$$F'(t) = -\frac{1}{4ir_j} S_\phi((H_j^2 + 4X_j^2)(a \circ A_j(t))).$$

Let $c_{a,t} = t \sup_{0 \leq s \leq t} |S_\phi((H_j^2 + 4X_j^2)(a \circ A_j(s)))|$, then

$$|S_\phi(a \circ A_j(t)) - S_\phi(f)| = |F(t) - F(0)| = |\int_0^t F'(s) ds| \leq \frac{c_{a,t}}{4r_j}.$$

$\square$

3.3. **Positivity.** Fix a subset $\mathcal{J} \subseteq \{1, \ldots, d\}$. We show that if for all $j \in \mathcal{J}$, r_j becomes large the distribution S_ϕ is close to a positive measure on $\Gamma \backslash \mathcal{G} / \prod_{i \notin \mathcal{J}} K_i$ (c.f., [7, Theorem 3.1]).

Proposition 3.3. *There are normalized functions $\Phi_\mathcal{J} \in L^2(Y)$, such that for any fixed $a \in C^\infty(Y)$ that is invariant under $\prod_{j \notin \mathcal{J}} K_j$,*

$$S_\phi(a) = \langle a\Phi_\mathcal{J}, \Phi_\mathcal{J} \rangle_Y + O_a\left(\max_{j \in \mathcal{J}} \{r_j^{-1/4}\} \right).$$

For the proof we will use the following lemma

Lemma 3.4. *Let $a \in C^\infty(Y)$, then*

$$\langle a\phi_n, \phi_m \rangle_Y = \langle a\phi_{n-e_j}, \phi_{m-e_j} \rangle_Y + O(\frac{N \|a\|_\infty + \left\|E_j^+ a\right\|_\infty}{r_j}),$$

where $N = \max(|n_j|, |m_j|)$

Proof. By definition

$$\langle a\phi_n, \phi_m \rangle_Y = \frac{\langle aE_j^+ \phi_{n-e_j}, \phi_m \rangle_Y}{(2ir_j + 2n_j - 1)}.$$

Replace $aE^+\phi_{n-e_j} = E_j^+(a\phi_{n-e_j}) - (E_j^+a)\phi_{n-e_j}$, and use the bound

$$|\langle (E^+a)\phi_{n-e_j}, \phi_m\rangle_Y| \leq \left\|E_j^+a\right\|_\infty$$

to get that

$$\langle a\phi_n, \phi_m\rangle_Y = \frac{\langle E_j^+(a\phi_{n-e_j}), \phi_m\rangle_Y}{(2ir_j + 2n_j - 1)} + O(\frac{\left\|E_j^+a\right\|_\infty}{r_j}).$$

Finally notice,

$$\begin{aligned}
\frac{\langle E_j^+(a\phi_{n-e_j}), \phi_m\rangle_Y}{(2ir_j + 2n_j - 1)} &= \frac{\langle a\phi_{n-e_j}, E_j^-\phi_m\rangle_Y}{(2ir_j + 2n_j - 1)} \\
&= \langle a\phi_{n-e_j}, \phi_{m-e_j}\rangle_Y(1 + O(\frac{N}{r_j})) \\
&= \langle a\phi_{n-e_j}, \phi_{m-e_j}\rangle_Y + O(\frac{N\left\|a\right\|_\infty}{r_j}).
\end{aligned}$$

□

For any subset $\mathcal{J}$, let $\mathbb{Z}^{\mathcal{J}} = \left\{n \in \mathbb{Z}^d | \forall j \notin \mathcal{J},\ n_j = 0\right\}$ and for any positive integer N let $Z_N^{\mathcal{J}} = \left\{n \in \mathbb{Z}^{\mathcal{J}} |\left\|n\right\|_\infty \leq N\right\}$. Define the function

$$\Phi_{N,\mathcal{J}} = (\frac{1}{2N+1})^{\frac{|\mathcal{J}|}{2}} \sum_{n \in Z_N^{\mathcal{J}}} \phi_n.$$

Proposition 3.5. *Fix a subset $\mathcal{J} \subseteq \{1, \ldots, d\}$ with $|\mathcal{J}| = J > 0$. Let $a \in C^\infty(Y)$ be invariant under $\prod_{j\notin\mathcal{J}} K_j$, and let $R = \min_{j\in\mathcal{J}} r_j$. Then*

$$S_\phi(a) = \langle a\Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y + O_a(\frac{N^2}{R}) + O_a(\frac{R^{J\epsilon}}{N^J}) + O_{a,\epsilon}(\frac{1}{R}),$$

(Taking $N \sim R^{1/3}$ and $\epsilon = \frac{1}{3} - \frac{1}{4J}$ gives the result of Proposition 3.3.)

Proof. Since a is invariant under $\prod_{j\notin\mathcal{J}} K_j$, its $\mathcal{K}$-Fourier decomposition is of the form $a = \sum_{n\in\mathbb{Z}^{\mathcal{J}}} a_n$. Let $a_\epsilon = \sum_{n\in Z^{\mathcal{J}}_{R^\epsilon}} a_n$, then $S_j(a) = S_j(a_\epsilon) + O_{a,\epsilon}(\frac{1}{R})$ and $\langle a\Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y = \langle a_\epsilon\Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y + O_{a,\epsilon}(\frac{1}{R})$, so it is sufficient to prove this for a_ϵ.

By repeating Lemma 3.4 at most N times for each $j \in \mathcal{J}$, we get

$$\langle a_\epsilon\Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y = (\frac{1}{2N+1})^J \sum_{n,m\in Z_N^{\mathcal{J}}} \langle a_\epsilon\phi_{(n-m)}, \phi_0\rangle_Y + O_a(\frac{N^2}{R}) =$$

$$= (\frac{1}{2N+1})^J \sum_{n\in Z_N^{\mathcal{J}}} \sum_{n+m\in Z_N^{\mathcal{J}}} \langle a_\epsilon\phi_m, \phi_0\rangle_Y + O_a(\frac{N^2}{R}).$$

Now note that $\langle a_\epsilon \phi_m, \phi_0\rangle_Y = 0$ unless $m \in Z^{\mathcal{J}}_{R^\epsilon}$. Consequently,

$$\begin{aligned}
&\langle a_\epsilon \Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y = \\
&= (\frac{1}{2N+1})^J \sum_{m\in Z^{\mathcal{J}}_{R^\epsilon}} \langle a_\epsilon \phi_m, \phi_0\rangle_Y \sharp\left\{n, n+m \in Z^{\mathcal{J}}_N\right\} + O_a(\frac{N^2}{R})
\end{aligned}$$

and

$$S_\phi(a_\epsilon) = (\frac{1}{2N+1})^J \sum_{m\in Z^{\mathcal{J}}_{R^\epsilon}} \langle a_\epsilon \phi_m, \phi_0\rangle_Y \sharp\left\{n \in Z^{\mathcal{J}}_N\right\},$$

We can thus bound the difference

$$\begin{aligned}
&|S_\phi(a_\epsilon) - \langle a_\epsilon \Phi_{N,\mathcal{J}}, \Phi_{N,\mathcal{J}}\rangle_Y| \ll_a \\
&\ll_a \left(\frac{1}{2N+1}\sum_{|m_j|\leq R^\epsilon} \sharp\left\{n_j \colon |n_j| \leq N < |n_j+m_j|\right\}\right)^J + O_a(\frac{N^2}{R}) \\
&= O_a(\frac{R^{J\epsilon}}{N^J}) + O_a(\frac{N^2}{R}).
\end{aligned}$$

□

4. Quantization procedure

We now wish to relate the micro local lift defined above, to the lift obtained via a quantization procedure. That is, for smooth functions $a \in C^\infty(TX)$ we assign operators $\mathrm{Op}(a)$ on $L^2(X)$, and for any Laplacian eigenfunction ϕ_k, we assign the distribution $a \mapsto \langle \mathrm{Op}(a)\phi_k, \phi_k\rangle$. We show that this functional is supported on $\Sigma(\lambda_k)$ and that after identifying $\Sigma(\lambda_k) \cong \Sigma(\mathbf{E}_k) \cong \Gamma\backslash\mathcal{G}$ this functional coincides with the functional S_{ϕ_k} defined above.

4.1. **Spherical Transforms.** Before proceeding with the construction, we digress and go over some of Helgason's results on hyperbolic harmonic analysis on $\mathrm{PSL}(2,\mathbb{R})$ that we will need [5]. In particular we will make use of the generalized spherical functions and spherical transforms. For the rest of this section we will concentrate on a single factor $G_j = \mathrm{PSL}(2,\mathbb{R})$, and for notational convenience the subscript j will be omitted.

For $n \in \mathbb{Z}$ let χ_n be the character of K given by $\chi_n(k_\theta) = e^{2in\theta}$ and complete it to a function on G by $\chi_n(pk) \equiv \chi_n(k)$. The generalized spherical functions $\Phi_{r,n} \in C^\infty(\mathbb{H})$ are given by

$$\Phi_{r,n}(z) = \int_K \varphi_r(k^{-1}z)\chi_n(k)dk,$$

where φ_r is the Laplacian eigenfunction $\varphi_r(x+iy) = y^{ir+\frac{1}{2}}$. Note that both $\Phi_{r,n}$ and $\Phi_{-r,n}$ are Laplacian eigenfunctions (with the same eigenvalue $\lambda = r^2 + \frac{1}{4}$) and they both satisfy $\Phi_{\pm r,n}(kz) = \chi_n(k)\Phi_{\pm r,n}(z)$. Therefore, $\Phi_{r,n}$ and $\Phi_{-r,n}$ differ by some constant, which can be computed explicitly as a quotient of Γ functions [5, Proposition 4.17]

$$\Phi_{r,n}(z) = \frac{P_n(2ir)}{P_n(-2ir)}\Phi_{-r,n}(z),$$

with $P_n(x) = \frac{\Gamma(\frac{x+1}{2}+|n|)}{\Gamma(\frac{x+1}{2})} = (\frac{x+1}{2})(\frac{x+1}{2}+1)\cdots(\frac{x+1}{2}+|n|-1)$.

Remark 4.1. For the interested reader, we remark that the spherical function, $\Phi_{r,n}$, can be expressed as a product of Γ functions and the $|n|$'th order Legandre function [17, Proposition 2.9]

$$\Phi_{r,n}(ie^t) = \frac{\Gamma(ir+\frac{1}{2}-|n|)}{\Gamma(ir+\frac{1}{2})}P^{|n|}_{ir-\frac{1}{2}}(\cosh(t)).$$

See also [5, equation 59] for another expression involving the hypergeometric function.

We will not make any direct use of these formulas, all we will use is the following asymptotic estimate on the spherical functions.

Lemma 4.1. *As $r \to \infty$*

$$|\Phi_{r,n}(z)| \ll \frac{1}{\sqrt{r}},$$

uniformly in any compact set not containing i.

Proof. Since $|\Phi_{r,n}(kz)| = |\Phi_{r,n}(z)|$, it is sufficient to show the bound for $\Phi_{r,n}(iy)$ for $y \neq 1$.

We can write, $\varphi_r(k_\theta(iy)) = e^{(2ir+1)\psi(y,\theta)}$, with

$$\psi(y,\theta) = \frac{1}{2}\log(\frac{y}{\sin^2(\theta)(y^2-1)+1}).$$

Now, for fixed $y \neq 1$ the function $\psi_y(\theta) = \psi(y,\theta)$ is a smooth function, its first derivative $\psi'_y(\theta)$ vanishes only when $\theta = \frac{\pi l}{2}$, $l \in \mathbb{Z}$ and the second derivative $\psi''_y(\frac{\pi l}{2}) = 1 - y^{\pm 2} \neq 0$ do not vanish at these points. We can now write $\Phi_{r,n}(iy) = \frac{1}{2\pi}\int_0^{2\pi} F_y(\theta)e^{ir\psi_y(\theta)}d\theta$, with $F_y(\theta) = e^{in\theta+\psi_y(\theta)}$ a smooth function. For such an integral by the method of stationary phase

$$\frac{1}{2\pi}\int_0^{2\pi} F_y(\theta)e^{ir\psi_y(\theta)}d\theta = O(\frac{1}{\sqrt{r}}).$$

The implied constant, can be given explicitly in terms of $\|F_y\|_\infty$, $\|F'_y\|_\infty$, $\|\psi'_y\|$, $\|\psi''_y\|$ and $\psi''_y(\frac{\pi l}{2})$, and hence can be chosen uniformly for any bounded segment not containing 1. ☐

Definition 4.2. For $n \in \mathbb{Z}$, let $C_n^\infty(\mathbb{H})$ denote the space of smooth compactly supported functions on $\mathbb{H}$ satisfying $f(kz) = \chi_n(k)f(z)$. Define the n-spherical transform on $C_n^\infty(\mathbb{H})$ by

$$\mathcal{S}_n(f)(r) = \int_{\mathbb{H}} f(z)\Phi_{r,-n}(z)dz.$$

(For $n = 0$, this is also known as the Selberg transform.)

We say that a holomorphic function $h(r)$ is of uniform exponential type R, if $\forall N \in \mathbb{N}$, $h(r) \ll_N \frac{e^{R|\mathrm{Im}(r)|}}{(1+|r|)^N}$. Let $PW(\mathbb{C})$ denote the space of holomorphic functions of uniform exponential type, and $PW_n(\mathbb{C})$ the subspace of holomorphic functions of uniform exponential type satisfying the functional equation $P_n(2ir)h(-r) = P_n(-2ir)h(r)$.

Proposition 4.3. *The n-spherical transform, $\mathcal{S}_n$, is a bijection of $C_n^\infty(\mathbb{H})$ onto $PW_n(\mathbb{C})$, with inverse transform given by*

$$\mathcal{S}_n^{-1}h(z) = \frac{1}{2\pi}\int_0^\infty h(r)\Phi_{-r,n}(z)r\tanh(\pi r)dr.$$

Moreover, if $h \in PW_n(\mathbb{C})$ is of uniform exponential type R, then $f = \mathcal{S}_n^{-1}h \in C_n^\infty(\mathbb{H})$ is supported in the disc $d(z,i) < R$.

Proof. For any $f \in C_c^\infty(\mathbb{H})$ it's Helgason-Fourier transform is given by

$$\tilde{f}(r,k) = \int_{\mathbb{H}} f(z)\varphi_{-r}(k^{-1}z)dz.$$

This transform is a bijection of $C_c^\infty(\mathbb{H})$ onto the space of holomorphic functions with uniform exponential type satisfying the functional equation

$$\int_K \varphi_r(k^{-1}z)\tilde{f}(r,k)dk = \int_K \varphi_{-r}(k^{-1}z)\tilde{f}(-r,k)dk.$$

The inverse transform is given by

$$f(z) = \frac{1}{2\pi}\int_0^\infty \int_K \tilde{f}(r,k)\varphi_r(k^{-1}z)r\tanh(\pi r)dr,$$

and if $\tilde{f}(r,k)$ is of uniform exponential type R, then $f(z)$ is supported on $d(i,z) \le R$ [5, Theorem 4.2]. The above proposition now follows directly from the identity (verified by a simple computation)

$$\forall f \in C_n^\infty(\mathbb{H}), \quad \tilde{f}(r,k) = \chi_n(k)\mathcal{S}_n f(-r).$$

☐

For $f \in C_c^\infty(\mathbb{H})$ let $L[f]\colon C^\infty(G/K) \to C^\infty(G)$ be the convolution operator defined by

$$L[f]u(g) = \int_{\mathbb{H}} f(g^{-1}w)u(w)dw.$$

Lemma 4.4. *Let $\phi \in C^\infty(\mathbb{H})$ be a Laplacian eigenfunction with eigenvalue $(r^2 + \frac{1}{4})$, and let $\phi_n \in C^\infty(G)$ satisfy $\phi_{n\pm1} = \frac{1}{2ir+1\pm2n}E^\pm\phi_n$ with $\phi_0(g) = \phi(g(i))$. Then for any $f \in C_n^\infty(\mathbb{H})$,*

$$L[f]\phi_0 = \mathcal{S}_n f(r)\phi_{-n}$$

Proof. First, by [5, Theorem 4.3], any Laplacian eigenfunction ϕ can be expressed as an integral $\phi(z) = \int_K \varphi_r(kz)dT(k)$ with respect to a suitable distribution on K. Hence, it is sufficient to show this in the special case where $\phi(z) = \varphi_r(kz)$ for arbitrary $k \in K$. Next, note that if $\tilde{\phi}(z) = \phi(kz)$ then $(L[f]\tilde{\phi})(g) = L[f]\phi(kg)$ and also $\tilde{\phi}_n(g) = \phi_n(kg)$ (because the left action of K commutes with $E^\pm$). Hence it is sufficient to show the equality only for $\phi(z) = \varphi_r(z)$. Finally, note that the functions $\phi_n(g) = \varphi_r(g(i))\chi_n(g)$ satisfy the above recursion relation. It thus remains to show that $L[f]\varphi_r(g) = S_n f(r)\varphi_r(g(i))\chi_{-n}(g)$.

Fix $g = p_z k$, then (after the change of variables $w \mapsto p_z w$)

$$L[f]\varphi_r(p_z k) = \int_{\mathbb{H}} f(k^{-1}w)\varphi_r(p_z w)dw.$$

The function φ_r satisfies $\varphi_r(p_z w) = \varphi_r(z)\varphi_r(w)$ so that

$$\begin{aligned} L[f]\varphi_r(p_z k) &= \varphi_r(z)\int_{\mathbb{H}} f(w)\varphi_r(kw)dw = \\ = \varphi_r(z)\tilde{f}(-r, k^{-1}) &= \mathcal{S}_n f(r)\chi_{-n}(k)\varphi_r(z). \end{aligned}$$

concluding the proof. □

4.2. **Quantization.** We now wish to relate the functionals S_{ϕ_k} to functionals obtained by diagonal matrix elements of some quantization procedure. For this we use a generalization of Zeldich's quantization procedure via Helgasons Fourier transform [16]. For any smooth function $a \in C^\infty(TX)$ we assign its quantization which is an integral operator $\mathrm{Op}(a)$ acting on $L^2(X)$. Recall the map $(z, \xi) \mapsto (p_z k, E(z, \xi))$ from TX to $\Gamma\backslash\mathcal{G} \times [0, \infty)^d$ and think of a function on TX as a function $a = a(p_z k, r)$ with the parametrization $r_j = \sqrt{E_j - \frac{1}{4}}$. Let

$$\tilde{a}(z, w) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{+d}}\int_{\mathcal{K}} a(p_z k, r)\left(\prod_{j=1}^d \varphi_{r_j}(k_j^{-1}w_j)r_j\tanh(\pi r_j)\right)drdk,$$

be the inverse Helgason-Fourier transform (in all of the (k_j, r_j) coordinates). We then define the operator $\mathrm{Op}(a)$ by the kernel

$$K(z,w) = \tilde{a}(z, p_z^{-1}w).$$

Since a is Γ invariant, this kernel satisfies $K(\gamma z, \gamma w) = K(z,w)$, and hence defines an operator on $L^2(X)$.

In the following lemma, we show that the Wigner distribution $a \mapsto \langle \mathrm{Op}(a)\phi_k, \phi_k \rangle_X$ is supported on $\Sigma(\lambda_k) \cong \Gamma\backslash\mathcal{G}$ and coincides there with S_{ϕ_k}.

Lemma 4.5. *Let $a = a(g,r) \in C^\infty(TX)$ be holomorphic of uniform exponential type (in the r_j variables) and satisfy that $\forall j = 1, \ldots, d$ the expression*

$$\int_{K_j} a(p_z k, r)\varphi_{r_j}(k_j^{-1} w_j) dk_j,$$

is invariant under the substitution $r_j \mapsto -r_j$. Then

$$\langle \mathrm{Op}(a)\phi_k, \phi_k \rangle = S_{\phi_k}(a(\cdot, r_k)).$$

Proof. We will give the proof in the special case where the function a is of the form $a(g,r) = a(g)h(r)$ with $h(r) = \prod_j h_j(r_j)$ and $a \in \mathcal{F}_n(Y)$ is of some fixed $\mathcal{K}$-type $n \in \mathbb{Z}^d$. (This is the only case that we will use, however, the general statement can be deduced by decomposing $a(g,r)$ into its $\mathcal{K}$-Fourier series.) For a of the above type, the functional equation is equivalent to the requirement that the functions $h_j \in PW_{n_j}(\mathbb{C})$. We can now write the kernel as

$$K_a(z,w) = a(p_z) \prod_{j=1}^{d} f_j(p_{z_j}^{-1} w_j),$$

with $f_j = \mathcal{S}_{n_j}^{-1}(h_j) \in C^\infty_{n_j}(\mathbb{H})$. In particular the operator $\mathrm{Op}(a)$ is given by a tensor product of convolution operators

$$\mathrm{Op}(a)\phi(z) = a(p_z) L[f_1] \otimes \cdots \otimes L[f_d]\phi(z).$$

Since $\phi_k(z)$ are joint eigenfunctions of all partial Laplacians, Lemma 4.4 (applied separately to each coordinate) implies

$$\mathrm{Op}(a)\phi_k(z) = a(p_z)h(r_k)\phi_{k,-n}(p_z).$$

We thus get that

$$\begin{aligned}\langle \mathrm{Op}(a)\phi_k, \phi_k\rangle_X &= \int_X \mathrm{Op}(a)\phi_k(z)\overline{\phi_k(z)}dz \\ &= \int_X a(p_z)h(r_k)\phi_{k,-n}(p_z)\overline{\phi_k(z)}dz \\ &= \int_Y a(g)h(r_k)\phi_{k,-n}(g)\overline{\phi_{k,0}(g)}dg \\ &= h(r_k)\langle a\phi_{k,-n}, \phi_{k,0}\rangle_Y = S_{\phi_k}(a(\cdot, r_k)).\end{aligned}$$

□

5. A Local Weyl's Law

We now give the proof of Theorem 1, showing that for large eigenvalues, on average, the distributions S_{ϕ_k} defined above converge to the volume measure of Y.

5.1. **A Trace Formula.** The main ingredient in the proof will be a trace formula, relating the sum over the eigenvalues to a sum over conjugacy classes in Γ. Recall the setting: $X = \Gamma\backslash\mathcal{H}$, $\{\phi_k\} \in C^\infty(X)$ is an orthonormal basis for $L^2(X)$ composed of joint Laplacian eigenfunctions (with eigenvalues $\lambda_{k,j} = (r_{k,j}^2 + \frac{1}{4})$ respectively) and S_{ϕ_k} the corresponding distributions.

For any $1 \leq j \leq d$, fix $f_j \in C^\infty_{n_j}(\mathbb{H})$, and let $h_j(r_j) = \mathcal{S}_{n_j} f_j \in PW_{n_j}(\mathbb{C})$ be the corresponding spherical transforms. Denote by $h(r) = \prod_j h_j(r_j)$ and by $f(z) = \prod_j f_j(z_j)$. For any $\gamma \in \Gamma$, let Γ_γ be the centralizer of γ in Γ and let $\mathcal{F}_\gamma \subseteq \mathcal{H}$ be a fundamental domain for Γ_γ.

Theorem 3. *For any observable $a \in \mathcal{F}_n(X)$*

$$\sum_k h(r_k)S_{\phi_k}(a) = \sum_{\{\gamma\}} \int_{\mathcal{F}_\gamma} a(p_z)f(p_z^{-1}\gamma z)dz,$$

where the right hand sum is over the conjugacy classes in Γ.

Remark 5.1. In the special case, when $n = 0$ and $a \equiv 1$ is the constant function, the terms $\int_{\mathcal{F}_\gamma} f(p_z^{-1}\gamma z)dz$ can be computed explicitly in terms of the Fourier transform of h, retrieving the Selberg trace formula.

Proof. Consider the operator $\mathrm{Op}(ah)$ given by the kernel

$$K(z, w) = a(p_z)f(p_z^{-1}w).$$

We can think of $\mathrm{Op}(ah)$ as an operator on $L^2(\Gamma\backslash\mathbb{H})$ with kernel given by

$$K_\Gamma(z,w)=\sum_\gamma K(z,\gamma w).$$

Write the trace of this operator in two different ways. First, since ϕ_k is an orthonormal basis for $L^2(X)$, by Lemma 4.5

$$\mathrm{Tr}(\mathrm{Op}(ah))=\sum_k \langle \mathrm{Op}(a)\phi_k,\phi_k\rangle_X=\sum_k h(r_k)S_{\phi_k}(a).$$

On the other hand, if $\mathcal{F}\subseteq\mathcal{H}$ is a fundamental domain for Γ then

$$\mathrm{Tr}(\mathrm{Op}(ah))=\int_{\mathcal{F}}K_\Gamma(z,z)dz=\sum_\gamma\int_{\mathcal{F}}K(z,\gamma z)dz.$$

Note that if $\gamma'=g^{-1}\gamma g$ are conjugated in Γ then

$$\int_{\mathcal{F}}K(z,\gamma' z)dz=\int_{\mathcal{F}}K(gz,\gamma gz)dz=\int_{g\mathcal{F}}K(z,\gamma z)dz.$$

We can thus write

$$\begin{aligned}\mathrm{Tr}(\mathrm{Op}(ah)) &= \sum_{\{\gamma\}}\int_{\mathcal{F}_\gamma}K(z,\gamma z)dz\\ &= \sum_{\{\gamma\}}\int_{\mathcal{F}_\gamma}a(p_z)f(p_z^{-1}\gamma z)dz\end{aligned}$$

where $\mathcal{F}_\gamma=\sum_{g\in\Gamma/\Gamma_\gamma}g\mathcal{F}_\gamma$ is the fundamental domain for Γ_γ. □

5.2. **Smoothing.** In order to use the trace formula to evaluate the sum $\sum_{k\in\mathcal{I}(\mathbf{L})}S_{\phi_k}(a)$, we need to approximate the window function by a smooth function admissible in the trace formula.

Definition 5.1. We say that a smooth function $h\in C^\infty(\mathbb{R})$ is δ-approximating the window function around $L\in[\frac12,\infty)$, if it satisfies for real $x>0$

$$|h(x)-\mathbb{1}_{[L-\frac12,L+\frac12]}(x)|=\begin{cases} O(\delta) & |x-L|\leq\frac12-\sqrt{\delta}\\ O(1) & \frac12-\sqrt{\delta}\leq|x-L|\leq\frac12\\ O_N(\delta(\frac{1}{|x-L|-1/2})^N) & |x-L|>\frac12\end{cases}$$

where $\mathbb{1}_{[\alpha,\beta]}$ is the indicator function of $[\alpha,\beta]$.

Let $\Theta(r)=\prod_j\mathbb{1}_{[-\frac12,\frac12]}(r_j)$ denote a window function around zero in $\mathbb{R}^d$. If $h_{L_j,\delta}$ are functions δ-approximating the window functions around L_j respectively, then their product $h_{\mathbf{L},\delta}(r)=\prod_j h_{L_j,\delta}(r_j)$ is a good approximation to the window function $\Theta(r-\mathbf{L})$ around $\mathbf{L}=(L_1,\dots L_j)$ in the following sense.

Proposition 5.2.
$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{1}{L_1\cdots L_d}\sum_{r_k\in\mathbb{R}^d} |h_{\mathbf{L},\delta}(r_k)-\Theta(r_k-\mathbf{L})| = O(\sqrt{\delta})$$

Proof. Appendix A, Proposition A.5. □

For $n\in\mathbb{Z}$, recall that $PW_n(\mathbb{C})$ is the space of holomorphic functions $h(x)$ of uniform exponential type, satisfying the functional equation $P_n(-2i)h(x) = P_n(2ix)h(-x)$ with
$$P_n(x) = (\frac{x+1}{2})(\frac{x+1}{2}+1)\cdots(\frac{x+1}{2}+|n|-1).$$
We will show that for any fixed $n\in\mathbb{Z}$, there are functions in $PW_n(\mathbb{C})$ that δ-approximate the window functions. For this we need the following lemma.

Lemma 5.3. *For fixed $n\in\mathbb{Z}$, there are holomorphic functions $F_\delta(x)$ satisfying*

- *The Fourier transform $\hat{F}_\delta\in C_c^\infty(\mathbb{R})$ is compactly supported.*
- $\forall|m|\leq|n|,\ F_\delta(\frac{im}{2})=1$
- *$F_\delta(x)=O(\delta)$, uniformly for real $x\in\mathbb{R}$.*

Proof. For $0<\delta<1$, let $G_\delta(x)=\sin(x/\delta)\prod_{1\leq|m|\leq|n|}\frac{2x-im}{x/\delta-m\pi}$. Then $G_\delta(\frac{im}{2})=0$, the derivative $G'_\delta(\frac{im}{2})\gg e^{\frac{m}{\delta}}\gg\frac{1}{\delta}$, and for real x the function $|G_\delta(x)|\leq 1$ is bounded. The function,
$$F_\delta(x)=\sum_{1\leq|m|\leq|n|}\frac{G_\delta(x)}{G'_\delta(im)(x-im)},$$
then satisfies the above properties [2]. □

Proposition 5.4. *For fixed $n\in\mathbb{Z}$, for any $L\geq\frac{1}{2}$ and $\delta>0$ there is a function $h_{L,\delta}\in PW_n(\mathbb{C})$ (with exponential type depending on δ but not on L), that is δ-approximating the window function around L.*

Proof. Fix a positive even holomorphic function $\rho\in PW_0(\mathbb{C})$ with Fourier transform $\hat{\rho}$ supported in $[-1,1]$ and $\hat{\rho}(0)=1$. For any $\delta>0$, let $\rho_\delta(x)=\frac{1}{\delta}\rho(\frac{x}{\delta})$ and define a smoothed window function by convolution with the window function $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$. Then the smoothed function $\mathbb{1}_\delta=\rho_\delta*\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$ satisfies
$$\mathbb{1}_\delta(x)=\left\{\begin{array}{cc} 1+O(\delta) & |x|\leq 1/2-\sqrt{\delta}\\ O((\frac{\delta}{|x|-1/2})^N) & |x|>\frac{1}{2}\end{array}\right.$$

[2] I thank Mikhail Sodin for showing me this construction.

We now want to deform the smoothed function $\mathbb{1}_\delta(x-L)$ into a function in $PW_n(\mathbb{C})$. For $n=0$ we can simply take

$$h_{L,\delta}(x) = \mathbb{1}_\delta(-x-L) + \mathbb{1}_\delta(x-L) \in PW_0(\mathbb{C}).$$

Otherwise, let $F_\delta(x)$ be as in Lemma 5.3, and define the function

$$h_{L,\delta}(x) = (1-F_\delta(x))\mathbb{1}_\delta(x-L) + \frac{P_n(2ix)}{P_n(-2ix)}(1-F_\delta(-x))\mathbb{1}_\delta(-x-L).$$

The function $h_{L,\delta}$ obviously satisfies the functional equation. The zeros of $1-F_\delta(x)$ cancel the poles of $\frac{P_n(2ix)}{P_n(-2ix)}$, and since the Fourier transform $\hat{F}_\delta$ is compactly supported, $h_{L,\delta}$ is of uniform exponential type (depending only on δ). It remains to show that it indeed approximates the window function.

First, note that for $x\in\mathbb{R}$ the function $|1-F_\delta(x)| = O(1)$ and $|\frac{P_n(-ix)}{P_n(ix)}| = 1$, so that $|h_{L,\delta}(x)| = O(1)$ is bounded. Now, for $|x-L| > 1/2$ we can bound

$$|h_{L,\delta}(x)| \leq |(1-F_\delta(x))||\mathbb{1}_\delta(x-L)| + |(1-F_\delta(-x))||\mathbb{1}_\delta(-x-L)|.$$

The function $|1-F_\delta(x)|$ is bounded and $\mathbb{1}_\delta(\pm x - L) \ll (\frac{\delta}{|x-L|-\frac{1}{2}})^N$, hence $|h_{L,\delta}(x)| \ll (\frac{\delta}{|x-L|-\frac{1}{2}})^N$.

Next, for $|x-L| \leq 1/2-\sqrt{\delta}$ we can bound $|h_{L,\delta}(x)-1|$ by

$$|\mathbb{1}_\delta(x-L)-1| + |F_\delta(x)||\mathbb{1}_\delta(x-L)| + |(1-F_\delta(-x))||\mathbb{1}_\delta(-x-L)|.$$

The first term is bounded by $O(\delta)$, the second term is bounded by $O(\delta)$ (because $F_\delta(x) = O(\delta)$), and the last term is also bounded by $O(\delta)$ (since $|x+L| \geq \frac{1}{2}+\sqrt{\delta}$). □

5.3. **Proof of Theorem 1.** Let $a\in C^\infty(Y)$. With out loss of generality we can assume that $\int_Y a = 0$ and that a is of some fixed $\mathcal{K}$-type n. We thus need to show that

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(L)}\sum_k S_{\phi_k}(a) = 0.$$

For $\delta>0$ and $j=1,\ldots,d$ let $h_{L_j,\delta}\in PW_{n_j}(\mathbb{C})$ (with exponential type depending on δ but not on L_j) δ-approximate the window function around L_j, and let $h_{\mathbf{L},\delta}(r) = \prod h_{L_j,\delta}(r_j)$.

Use the trace formula for $h_{\mathbf{L},\delta}(r)$ to get

$$\sum_k h_{\mathbf{L},\delta}(r_k)S_{\phi_k}(a) = \sum_{\{\gamma\}}\int_{\mathcal{F}_\gamma} a(p_z)(\prod_{j=1}^d f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j))dz,$$

where $f_{L_j,\delta} = \mathcal{S}_{n_j}^{-1} h_{L_j,\delta} \in C_{n_j}^\infty(\mathbb{H})$ and $\mathcal{F}_\gamma \subseteq \mathbb{H} \times \cdots \times \mathbb{H}$ is the fundamental domain for Γ_γ.

First notice that the conjugacy class of the identity does not contribute anything. To see this write its contribution as

$$\int_{\mathcal{F}} a(p_z)(\prod_{j=1}^d f_{L_j,\delta}(p_{z_j}^{-1} z_j))dz = (\prod_{j=1}^d f_{L_j,\delta}(i)) \int_{\mathcal{F}} a(p_z)dz.$$

If there is some $n_j \neq 0$ then $f_{L_j,\delta}(i) = 0$. Otherwise a is $\mathcal{K}$ invariant and $\int_{\mathcal{F}} a(p_z)dz = \int_Y a(g)dg = 0$.

Next, recall that the functions $f_{L_j,\delta}$ are compactly supported so we can replace the noncompact domains $\mathcal{F}_\gamma$ by compact domains of the form $\tilde{\mathcal{F}}_\gamma = \{z \in \mathcal{F}_\gamma \colon d(z_j, \gamma_j z_j) < M\}$ for some constant $M = M(\delta)$ depending on δ. Denote by $l_{\gamma_j} = \inf_{\mathbb{H}} d(z_j, \gamma_j z_j)$ and note that there can be only a finite number of conjugacy classes satisfying that $\max_j l_{\gamma_j} \leq M$, hence, there are only a finite number of conjugacy class that contribute to the sum (the number depending again on δ but not on $\mathbf{L}$).

We now use the inverse transform to estimate the size of $f_{L_j,\delta}$,

$$f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j) = \frac{1}{2\pi}\int_0^\infty h_{L_j,\delta}(r)\Phi_{r,n_j}(p_{z_j}^{-1}\gamma_j z_j) r_j \tanh(\pi r_j) dr_j.$$

Since we assume Γ is irreducible and co-compact for any nontrivial conjugacy classes $\{\gamma\}$, we know that γ_j is either hyperbolic or elliptic. If γ_j is hyperbolic we can use Lemma 4.1 to bound $\Phi_{r_j,n_j}(p_{z_j}^{-1}\gamma_j z_j) \ll_\delta \frac{1}{\sqrt{r_j}}$ uniformly in the annulus $l_{\gamma_j} \leq d(p_{z_j}^{-1}\gamma z_j, i) \leq M$. We thus get the bound

$$f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j) \ll_\delta \int_0^\infty |h_{L_j,\delta}(r_j)|\sqrt{r_j} dr_j \ll_\delta \sqrt{L_j}.$$

In the case where γ_j is elliptic, for any $\epsilon > 0$ as before we can bound $f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j) \ll_{\epsilon,\delta} \sqrt{L_j}$, uniformly in the annulus $\epsilon \leq d(p_{z_j}^{-1}\gamma z_j, i) \leq M$. For $d(p_{z_j}^{-1}\gamma z_j, i) < \epsilon$ (i.e., in an ϵ-neighborhood of the fixed point of γ_j) we use the trivial bound $f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j) \ll L_j$ (coming from the estimate $\phi_{r,n_j}(p_{z_j}^{-1}\gamma_j z_j) = O(1)$).

Plugging these estimates in the integral, for strictly hyperbolic conjugacy classes

$$\int_{\tilde{\mathcal{F}}_\gamma} a(p_z)(\prod_{j=1}^d f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j))dz = O_\delta(\sqrt{L_1 \cdots L_d}),$$

and for mixed conjugacy classes (where some of the elements are elliptic)

$$\int_{\tilde{\mathcal{F}}_\gamma} a(p_z)(\prod_{j=1}^d f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j))dz = O_{\delta,\epsilon}(\sqrt{L_1\cdots L_d}) + O(\epsilon L_1\cdots L_d).$$

This is true for any $\epsilon > 0$, hence for any conjugacy class

$$\int_{\tilde{\mathcal{F}}_\gamma} a(p_z)(\prod_{j=1}^d f_{L_j,\delta}(p_{z_j}^{-1}\gamma_j z_j))dz = o(L_1\cdots L_d),$$

and thus for the whole sum

$$\sum_k h_{\mathbf{L},\delta}(r_k)S_{\phi_k}(a) = o(L_1\cdots L_d).$$

Taking the limit, recalling that $N(\mathbf{L}) \gg L_1\cdots L_d$ (Proposition A.2) we get that

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_k h_{\mathbf{L},\delta}(r_k)S_{\phi_k}(a) = 0.$$

The contribution from the exceptional eigenfunctions, where $r_{k,j}$ is imaginary, is negligible (see Lemma A.3), hence

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_{r_k\in\mathbb{R}^d} h_{\mathbf{L},\delta}(r_k)S_{\phi_k}(a) = 0.$$

Because $h_{L_j,\delta}(r_j)$ are δ-approximating the window functions around L_j, by Proposition 5.2

$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_{r_k\in\mathbb{R}^d} |h_{\mathbf{L},\delta}(r_k) - \Theta(r_k-\mathbf{L})| = O(\sqrt{\delta}),$$

implying that

$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_{k\in\mathcal{I}(L)} S_{\phi_k}(a) = O(\sqrt{\delta}).$$

This is true for any $\delta > 0$, hence

$$\lim_{\|\mathbf{L}\|\to\infty} \frac{1}{N(\mathbf{L})}\sum_k S_{\phi_k}(a) = 0.$$

Appendix A. Counting Eigenvalues

Let $X = \Gamma\backslash\mathcal{H}$ be a compact locally symmetric space with $\mathcal{H} = \mathbb{H}\times\cdots\times\mathbb{H}$ a product of d hyperbolic planes, $\mathcal{G} = \mathrm{PSL}(2,\mathbb{R})^d$ the group of isometries, and $\Gamma\subseteq\mathcal{G}$ an irreducible co-compact lattice. Let $\{\phi_k\}$ be a basis for $L^2(X)$ composed of Laplacian eigenfunctions (with eigenvalues $\lambda_{k,j} = \frac{1}{4}+r_{k,j}^2$). For $\mathbf{L} = (L_1,\dots,L_d)\in[\frac{1}{2},\infty)^d$ let

$$N(\mathbf{L}) = \sharp\left\{k\colon\ \|r_k-\mathbf{L}\|_\infty\le\frac{1}{2}\right\}.$$

Theorem A. *As* $\mathbf{L}\to\infty$*,*

$$L_1\cdots L_d \ll N(\mathbf{L}) \ll L_1\cdots L_d$$

Remark A.1. This theorem can be deduced from the analysis of Duistermaat, Kolk and Varadajan on the spectrum of compact locally symmetric spaces [3, Theorem 7.3]. However, for the sake of completeness, we will include here a self contained proof of this result.

In order to prove Theorem A, we will prove separately the upper and lower bounds. For the upper bound, we consider the number of eigenvalues in a scaled window $N(\mathbf{L},\epsilon) = \sharp\left\{k\colon\ \|r_k-\mathbf{L}\|_\infty\le\frac{\epsilon}{2}\right\}$.

Proposition A.1. *There is a constant $c_1>0$ such that for every $\epsilon>0$*

$$\limsup_{\|\mathbf{L}\|\to\infty}\frac{N(\mathbf{L},\epsilon)}{L_1\cdots L_d}\le c_1\epsilon^d$$

In particular for $\epsilon=1$, $N(\mathbf{L})\ll L_1\cdots L_d$. Now for the lower bound:

Proposition A.2. *There is a constant $c_2>0$, such that*

$$\liminf_{\|\mathbf{L}\|\to\infty}\frac{N(\mathbf{L})}{L_1\cdots L_d}\ge c_2$$

A.1. **Selberg Trace Formula.** The main tool we use for the proof of Propositions A.1 and A.2 is the Selberg trace formula (see [4, Sections 1-6] for the full derivation of the trace formula in this setting).

For any $\gamma\in\Gamma$ denote by $\{\gamma\}\in\Gamma^\sharp$ its conjugacy class, by Γ_γ its centralizer in Γ, and by $\mathcal{G}_\gamma$ it centralizer in $\mathcal{G}$. Let $h_j(r_j)\in C^\infty(\mathbb{R})$ be even and holomorphic in the strip $|\mathrm{Im}(r_j)|\le C$ for some fixed $C>\frac{1}{2}$. For any conjugacy class $\{\gamma\}\in\Gamma^\sharp$, let $c_\gamma = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$. Recall that for any $\gamma\in\Gamma$ its projections to the different factors are either hyperbolic, $\gamma_j\sim\begin{pmatrix} e^{l_j/2} & 0\\ 0 & e^{-l_j/2}\end{pmatrix}$, or elliptic $\gamma_j\sim\begin{pmatrix}\cos\theta_j & \sin\theta_j\\ -\sin\theta_j & \cos\theta_j\end{pmatrix}$. Define the functions $\tilde{h}_j(\gamma_j)$ by

$$\tilde{h}_j(\gamma_j) = \frac{\hat{h}(l_j)}{\sinh(l_j/2)},$$

when γ_j is hyperbolic, and

$$\tilde{h}_j(\gamma_j) = \frac{1}{\sin\theta_j}\int_{-\infty}^{\infty}\frac{\cosh[(\pi-2\theta_j)r]}{\cosh(\pi r)}h(r)dr$$

when γ_j is elliptic. The Selberg trace formula, applied to the product $h(r)=\prod h_j(r_j)$, then takes the form

$$\sum_k h(r_k) = \prod_j\left(\frac{1}{4\pi}\int_{\mathbb{R}} h_j(r_j)r_j\tanh(\pi r_j)dr_j\right) + \sum_{\{\gamma\}} c_\gamma\tilde{h}(\gamma),$$

where the right hand sum is over the nontrivial conjugacy classes $\{\gamma\}\in\Gamma^\sharp$ and $\tilde{h}(\gamma)=\prod\tilde{h}_j(\gamma_j)$.

A.2. **Exceptional eigenfunctions.** Recall that an exceptional eigenfunction is an eigenfunction for which some of the eigenvalues are small $0<\lambda_{k,j}<\frac{1}{4}$ (or equivalently $r_{k_j}\in i(0,\frac{1}{2})$). We now do a separate treatment of the contribution of these eigenfunctions to the trace formula. We show that the exceptional eigenfunctions are of density zero, so that their contribution to the trace formula can be neglected.

For any subset $\mathcal{J}\subset\{1,\dots,d\}$, denote by $\mathcal{I}(\mathcal{J})$ the set of (exceptional) eigenfunctions ϕ_k for which the j'th partial eigenvalue is small for $j\in\mathcal{J}$ (and not small otherwise). That is

$$\mathcal{I}(\mathcal{J}) = \left\{k\colon \forall j\in\mathcal{J}, \lambda_{k_j}<\frac{1}{4},\ \forall j\notin\mathcal{J}, \lambda_{k,j}\geq\frac{1}{4}\right\}.$$

Also denote by

$$\mathcal{I}(\mathcal{J},\mathbf{L}) = \left\{k\colon \forall j\in\mathcal{J}, \lambda_{k_j}<\frac{1}{4},\ \forall j\notin\mathcal{J}, |r_{k,j}-L_j|\leq\frac{1}{2}\right\},$$

and let $N(\mathcal{J},\mathbf{L})=\sharp\mathcal{I}(\mathcal{J},\mathbf{L})$.

Lemma A.3. *For any nonempty subset $\mathcal{J}\subset\{1,\dots,d\}$,*

$$\lim_{\|\mathbf{L}\|\to\infty}\frac{N(\mathcal{J},\mathbf{L})}{L_1\cdots L_d}=0$$

Proof. We will prove this for $\mathcal{J}=\{1,\dots,s-1\}$ (the proof is analogous for any other subset). For any $T>0$ define the function

$$h_{T,\mathbf{L}}(r) = e^{-\frac{T}{2}(r_1^2+\cdots+r_{s-1}^2)}\prod_{j=s}^{d}(e^{-\frac{(r_j-L_j)^2}{2}}+e^{-\frac{(r_j+L_j)^2}{2}})^2.$$

When the coordinates r_j are real or imaginary, the function $h_{T,\mathbf{L}}(r)$ is a positive real function. Moreover, if we assume that r_j is imaginary for $1\leq j\leq s-1$, and that $|r_j-L_j|\leq\frac{1}{2}$ for $s\leq j\leq d$, then

$h_{T,\mathbf{L}}(r) > e^{-\frac{d-s+1}{2}} \geq \frac{1}{e}$ is uniformly bounded away from zero. We can thus bound

$$N(\mathcal{J},\mathbf{L}) \ll \sum_k h_{T,\mathbf{L}}(r_k).$$

Now, plugging the functions $h_{T,\mathbf{L}}$ in the Selberg trace formula we get

$$N(\mathcal{J},\mathbf{L}) \ll \int_{\mathbb{R}^d} h_{T,\mathbf{L}}(r) \prod_j r_j \tanh(\pi r_j) dr + \sum_{\{\gamma\}} c_\gamma \tilde{h}_{T,\mathbf{L}}(\gamma).$$

The contribution from the nontrivial conjugacy classes is bounded by some constant depending on T but not on $\mathbf{L}$, while the integral is bounded by $O(\frac{L_s \cdots L_d}{T})$. Dividing by $L_1 \cdots L_d$ and taking $\mathbf{L} \to \infty$ we get

$$\limsup_{\|\mathbf{L}\| \to \infty} \frac{N(\mathcal{J},\mathbf{L})}{L_1 \cdots L_d} = O(\frac{1}{T}).$$

Now take $T \to \infty$ to conclude the proof. □

Lemma A.4. *Let $h_j \in C^\infty(\mathbb{R})$ be holomorphic and satisfy $|h_j(r_j)| \ll \frac{1}{|r_j|^3}$ uniformly in the strip $|\mathrm{Im}(r)| \leq \frac{1}{2}$. Let $\mathcal{I}_s$ denote the set of exceptional eigenfunctions. Define $h_{\mathbf{L}}(r) = \prod_j h_j(r_j - L_j)$, then*

$$\lim_{\|\mathbf{L}\| \to \infty} \frac{1}{L_1 \cdots L_d} \sum_{k \in \mathcal{I}_s} h_{\mathbf{L}}(r_k) = 0$$

Proof. It is sufficient prove this when taking the sum over $k \in \mathcal{I}(\mathcal{J})$ for an arbitrary nonempty subset $\mathcal{J} \subset \{1, \dots d\}$. We will show this for $\mathcal{J} = \{s+1, \dots, d\}$ (the proof for any other set is analogous).

We can write the corresponding sum as

$$\sum_{k \in \mathcal{I}(\mathcal{J})} h_{\mathbf{L}}(r_k) = \sum_{\mathbf{M} \in \mathbb{Z}^s} \sum_{k \in \mathcal{I}(\mathcal{J}, \mathbf{L} - \mathbf{M})} h_{\mathbf{L}}(r_k),$$

where we embed $\mathbf{M} = (M_1, \dots, M_s, 0, \dots, 0) \subset \mathbb{Z}^d$ in the natural way.

For fixed $k \in \mathcal{I}(\mathcal{J}, \mathbf{L} - \mathbf{M})$ and any $j \notin \mathcal{J}$, $|r_{k,j} - L_j| \geq |M_j| - \frac{1}{2}$. We can thus deduce that $|h_j(r_{k,j} - L_j)| = O((\frac{1}{M_j^3}))$. For $j \in \mathcal{J}$, we have $r_{k,j} - L_j = i\tilde{r}_{k,j} - L_j$ with $\tilde{r}_{k,j} < \frac{1}{2}$ bounded. Consequently, $|h_j(r_{k,j} - L_j)| = O(\frac{1}{L_j}) = O(1)$ is bounded. We thus have

$$\frac{1}{L_1 \cdots L_d} \sum_{k \in \mathcal{I}(\mathcal{J})} h_{\mathbf{L}}(r_k) \ll \frac{1}{L_1 \cdots L_d} \sum_{\mathbf{M} \in \mathbb{Z}^s} \frac{N(\mathcal{J}, \mathbf{L} - \mathbf{M})}{\prod_{j \notin \mathcal{J}} \max(M_j^3, 1)}.$$

On the other hand, from the previous lemma, for every $\epsilon > 0$ there is $R > 0$ so that for $\|\mathbf{L}\| > R$, $N(\mathcal{J}, \mathbf{L}) \leq \epsilon L_1 \cdots L_d$. Separate the sum into two terms, the first a finite sum over the terms $\mathbf{M}$ for which

$\|\mathbf{L}-\mathbf{M}\| \leq R$, and the second when $\|\mathbf{L}-\mathbf{M}\| > R$. The first term is bounded by

$$\frac{\sharp\{\mathbf{M}\colon \|\mathbf{M}\| \leq R\} \cdot \max\{N(\mathcal{J},\mathbf{M})\colon \|\mathbf{M}\| \leq R\}}{L_1 \cdots L_d} = O_R(\frac{1}{L_1 \cdots L_d}),$$

and the second by

$$\begin{aligned}\frac{\epsilon}{L_1 \cdots L_d} \sum_{\|\mathbf{L}-\mathbf{M}\|>R} \frac{\prod_j |L_j - M_j|}{\prod_j \max(|M_j|^3, 1)} &\leq \epsilon \sum_{\mathbf{M}} \prod_j \frac{1}{\max(M_j^2, 1)} \\ &\leq \epsilon(1 + \sum_{M \neq 0} \frac{1}{M^2})^s.\end{aligned}$$

Therefore, when taking $\mathbf{L} \to \infty$

$$\limsup_{\|\mathbf{L}\| \to \infty} \frac{1}{L_1 \cdots L_d} \sum_{k \in \mathcal{I}_s} h_{\mathbf{L}}(r_k) = O(\epsilon)$$

and taking $\epsilon \to 0$ concludes the proof.

□

A.3. **Proof of Proposition A.1.** Fix a positive even smooth function $h \in C^\infty(\mathbb{R})$, with Fourier transform $\hat{h}$ compactly supported. For each $L_j \geq \frac{1}{2}, \epsilon > 0$ let $h_j(r_j) = h_{L_j,\epsilon}(r_j) = h(\frac{r_j - L_j}{\epsilon}) + h(\frac{-r_j - L_j}{\epsilon})$.

For $r_j \in \mathbb{R}$ real, the function $h_{L_j,\epsilon}$ is a positive function, and for $|r_j - L_j| \leq \frac{\epsilon}{2}$ it is uniformly bounded away from 0. We can thus bound

$$N(\mathbf{L}, \epsilon) \ll \sum_{r_k \in \mathbb{R}^d} \prod h_{L_j,\epsilon}(r_{k,j}).$$

From the previous lemma, the contributions of the exceptional eigenfunctions can be bounded by $o(L_1 \cdots L_d)$ hence

$$N(\mathbf{L}, \epsilon) \ll \sum_k \prod h_{L_j,\epsilon}(r_{k,j}) + o(L_1 \cdots L_d).$$

For the full sum, by the Selberg trace formula, we get

$$\begin{aligned}\sum_k \prod h_{L_j,\epsilon}(r_{k,j}) &= \prod_{j=1}^d \left(\int_{\mathbb{R}} h_{L_j,\epsilon}(r_j) r_j \tanh(\pi r_j) dr_j\right) \\ &+ \sum_{\{\gamma\}} c_\gamma \prod_{j=1}^d \tilde{h}_{L_j,\epsilon}(\gamma_j).\end{aligned}$$

Notice that the Fourier transform $\hat{h}_{L_j,\epsilon}(t) = 2\epsilon \cos(L_j t) \hat{h}(\epsilon t)$, and since we assumed $\hat{h}$ compactly supported, there are only a finite number (depending on ϵ) of nontrivial conjugacy classes contributing to the

sum. Each contribution is bounded by some constant (not depending on $\mathbf{L}$), so that

$$N(\mathbf{L},\epsilon) \ll \prod_{j=1}^{d}\left(\int_{\mathbb{R}} h_{L_j,\epsilon}(r_j) r_j \tanh(\pi r_j) dr_j\right) + o(L_1\cdots L_d) + O_\epsilon(1).$$

We can estimate the integral

$$\int_{\mathbb{R}} h_{L_j,\epsilon}(r_j) r_j \tanh(\pi r_j) dr_j \ll \int_0^\infty h(\frac{r-L_j}{\epsilon}) r dr \ll L_j\epsilon,$$

to get the bound

$$N(\mathbf{L},\epsilon) \ll \epsilon^d L_1\cdots L_d + o(L_1\cdots L_d) + O_\epsilon(1).$$

Now divide by $L_1\cdots L_d$ and take $\mathbf{L}\to\infty$ to get

$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{N(\mathbf{L},\epsilon)}{L_1\cdots L_d} \ll \epsilon^d.$$

A.4. **Smoothing.** We now approximate the window function by a smoothed function admissible in the Selberg trace formula. Recall that a smooth function $h\in C^\infty(\mathbb{R})$ is δ-approximating the window function around $L\in\mathbb{R}$, if it satisfies for real $x>0$

$$|h(x) - \mathbb{1}_{[L-\frac{1}{2},L+\frac{1}{2}]}(x)| = \begin{cases} O(\delta) & |x-L| \le \frac{1}{2}-\sqrt{\delta} \\ O(1) & \frac{1}{2}-\sqrt{\delta} \le |x-L| \le \frac{1}{2} \\ O_N(\delta(\frac{1}{|x-L|-1/2})^N) & |x-L| > \frac{1}{2} \end{cases}$$

Proposition A.5. *Let $h_{L_j,\delta}\in C^\infty(\mathbb{R})$ be δ-approximating the window functions around L_j respectively. Let $h_{\mathbf{L},\delta}(r) = \prod_j h_{L_j,\delta}(r_j)$ be the corresponding approximation of the window function $\Theta(r-\mathbf{L})$ around $\mathbf{L}$. Then*

$$\limsup_{\|\mathbf{L}\|\to\infty} \frac{1}{L_1\cdots L_d} \sum_{r_k\in\mathbb{R}^d} |h_{\mathbf{L},\delta}(r_k) - \Theta(r_k-\mathbf{L})| = O(\sqrt{\delta})$$

Proof. We can write the sum differently as

$$\sum_{0\neq\mathbf{M}\in\mathbb{Z}^d} \sum_{k\in\mathcal{I}(\mathbf{L}-\mathbf{M})} |\prod_{j=1}^{d} h_{L_j,\delta}(r_{k,j})| + \sum_{k\in\mathcal{I}(\mathbf{L})} |\prod_{j=1}^{d} h_{L_j,\delta}(r_{k,j}) - 1|,$$

In the first sum, for $k\in\mathcal{I}(\mathbf{L}-\mathbf{M})$, we can bound $h_{L_j,\delta}(r_{k,j}) = O_N(\delta(M_j)^{-N})$ if $M_j\neq 0$ and $h_{L_j,\delta}(r_{k,j}) = O(1)$ otherwise. We then

evaluate $\sharp\mathcal{I}(\mathbf{L}-\mathbf{M}) = O((L_1-M_1)\cdots(L_d-M_d))$ (Proposition A.1) and get a bound on the first sum of order

$$\sum_{\mathbf{M}\neq 0}(\prod_{j=1}^d \min(\delta\frac{L_j-M_j}{M_j^N},1)) = O(\delta L_1\cdots L_d).$$

We now evaluate the second sum. For any $\epsilon>0$ denote by

$$\mathcal{I}(L,\epsilon) = \left\{k\colon\ \|r_k-\mathbf{L}\|_\infty \leq \frac{\epsilon}{2}\right\}.$$

We can separate the sum over $\mathcal{I}(\mathbf{L})$ to a sum over $\mathcal{I}(\mathbf{L},1-\sqrt{\delta})$ and the rest. For $k\in\mathcal{I}(\mathbf{L},1-\sqrt{\delta})$ we can evaluate $h_{L_j,\delta}=1+O(\delta)$, and the number of such eigenvalues is bounded by $N(\mathbf{L}) = O(L_1\cdots L_d)$ implying that

$$\sum_{k\in\mathcal{I}(\mathbf{L},1-\sqrt{\delta})}(\prod_{j=1}^d h_{L_j,\delta}(r_{k,j})-1) = O(\delta L_1\cdots L_d).$$

We are left with the sum over $\mathcal{I}(\mathbf{L})\setminus\mathcal{I}(\mathbf{L},1-\sqrt{\delta})$. This set can be covered by $O(\delta^{-\frac{d-1}{2}})$ boxes of size $\delta^{\frac{d}{2}}$. Since the number of eigenvalues in each such box is bounded by $O(\delta^{\frac{d}{2}}L_1\cdots L_d)+o(L_1\cdots L_d)$ (Proposition A.1) we can bound

$$\sharp(\mathcal{I}(\mathbf{L})\setminus\mathcal{I}(\mathbf{L},1-\sqrt{\delta})) = O(\sqrt{\delta}L_1\cdots L_d)+o(L_1\cdots L_d).$$

Since the functions $h_{L_j,\delta}=O(1)$ are bounded, this is also the bound for the remaining sum.

We have thus seen that the difference

$$\sum_{r_k\in\mathbb{R}^d}|h_{\mathbf{L},\delta}(r_k)-\Theta(r_k-\mathbf{L})| = O(\sqrt{\delta}L_1\cdots L_d)+o(L_1\cdots L_d).$$

Dividing by $L_1\cdots L_d$ and taking $\|\mathbf{L}\|\to\infty$ concludes the proof. $\square$

A.5. **Proof of Proposition A.2.** Fix a positive even holomorphic function $\rho\in PW_0(\mathbb{C})$ with Fourier transform $\hat{\rho}$ supported in $[-1,1]$ and $\hat{\rho}(0)=1$. For any $\delta>0$, let $\rho_\delta(x)=\frac{1}{\delta}\rho(\frac{x}{\delta})$ and define a smoothed window function by convolution $\mathbb{1}_\delta=\rho_\delta*\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$. For $j=1,\ldots,d$ the function

$$h_{L_j,\delta}(r_j)=\mathbb{1}_\delta(-r_j-L_j)+\mathbb{1}_\delta(r_j-L_j),$$

is δ-approximating the window function around L_j, and the function $h_{\mathbf{L},\delta}(r)=\prod_j h_j(L_j,\delta)(r_j)$ is admissible in the Selberg trace formula.

Hence,

$$\sum_k h_{\mathbf{L},\delta}(r_k) = \prod_{j=1}^{d} \left(\int_{\mathbb{R}} h_{L_j,\delta}(r_j) r_j \tanh(\pi r_j) dr_j \right) + \sum_{\{\gamma\}} c_\gamma \tilde{h}_{\mathbf{L},\delta}(\gamma).$$

As in the proof of Proposition A.1, the contribution of the nontrivial conjugacy classes is bounded by $O_\delta(1)$. We can bound the integrals

$$\int_{\mathbb{R}} h_{L_j,\delta}(r_j) r_j \tanh(\pi r_j) dr_j \gg L_j,$$

uniformly for $L_j \geq \frac{1}{2}$. Therefore, there is $c > 0$ such that

$$\sum_k h_{\mathbf{L},\delta}(r_k) \geq cL_1 \cdots L_d + O_\delta(1).$$

The contribution of the exceptional eigenfunctions is $o(L_1 \cdots L_d)$, and by Proposition A.5 the contribution of all other eigenvalues differ from $N(L)$ by $O(\sqrt{\delta} L_1 \cdots L_d) + o(L_1 \cdots L_d)$. We can deduce that

$$\frac{N(L)}{L_1 \cdots L_d} \geq c + O(\sqrt{\delta}) + O_\delta\left(\frac{1}{L_1 \cdots L_d}\right) + o(1)$$

Taking $\mathbf{L} \to \infty$, and then $\delta \to 0$ concludes the proof.

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